Physics 310/610 – Cosmology

Solution Set T

1. **[15] One things we haven't discussed, up to now, is pressure. All particles in the universe contribute to the pressure, but neutrinos are irrelevant because they don't interact with anything. We will look first at recombination, around $z = 1090$, $k_B T = 0.256$ eV.**

   (a) [6] The current density of baryons and dark matter are about $\rho_{b0} = 4.196\times10^{-28}\ \text{kg/m}^3$ and $\rho_{d0} = 2.235\times10^{-27}\ \text{kg/m}^3$. What was their density at recombination? What was the number density, assuming the baryons are hydrogen atoms, and dark matter has a mass of $m_d = 10^3\ \text{GeV}/c^2$?

Recombination occurred at about $z = 1090$, Since both baryons and dark matter are assumed to be non-relativistic, their density is proportional to $a^{-3} \propto (1+z)^3$, so we have

$$\rho_b = \rho_{b0}(1+z)^3 = \left(4.196\times10^{-28}\ \text{kg/m}^3\right)(1091)^3 = 5.45\times10^{-19}\ \text{kg/m}^3\ ,$$

$$\rho_d = \rho_{d0}(1+z)^3 = \left(2.235\times10^{-27}\ \text{kg/m}^3\right)(1091)^3 = 2.90\times10^{-18}\ \text{kg/m}^3\ .$$

The ^{1}H atom has an atomic "weight" of

$$m_H = 1.0078\ \text{u} = 1.0078\times1.6605\times10^{-27}\ \text{kg} = 1.673\times10^{-27}\ \text{kg}.$$

The dark matter has a mass of

$$m_d = 10^3\ \text{GeV}/c^2 = \frac{\left(10^{12}\ \text{eV}\right)\left(1.602\times10^{-19}\ \text{J/eV}\right)}{\left(2.998\times10^8\ \text{m/s}\right)^2} = 1.78\times10^{-24}\ \text{kg}\ .$$

The number density, therefore, is about

$$n_H = \frac{5.45\times10^{-19}\ \text{kg/m}^3}{1.673\times10^{-27}\ \text{kg}} = 3.257\times10^8\ \text{m}^{-3},$$

$$n_d = \frac{2.90\times10^{-18}\ \text{kg/m}^3}{1.78\times10^{-24}\ \text{kg}} = 1.63\times10^6\ \text{m}^{-3}\ .$$

   (b) [3] Assuming cold dark matter is the same temperature as baryons, they should each contribute a pressure of approximately $P = k_B nT$. In fact, the cold dark matter particles are almost certainly colder. Which contribution is more important? Calculate $P = k_B nT$ for the more important component.

The one with the higher number density contributes more. So it is the baryons, not the dark matter, that contribute more, since their number density is 200 times as high. We therefore have

$$P = k_B nT = \left(3.257\times10^{8}\ \text{m}^{-3}\right)\left(0.256\ \text{eV}\right) = 8.34\times10^{7}\ \text{eV/m}^3 = 1.336\times10^{-11}\ \text{N/m}^2\ .$$

(c) [4] In addition the radiation itself generates some pressure, given by $P = \frac{1}{3}u$, where *u* is the energy density. Calculate the radiation pressure. Of the three components, which is most important?

We simply use the various formulas we have, which give

$$P = \frac{1}{3}\cdot\frac{\pi^2\left(k_BT\right)^4}{15\left(\hbar c\right)^3} = \frac{\pi^2\left(0.256\ \text{eV}\right)^4}{45\left(6.582\times10^{-16}\ \text{eV}\cdot\text{s}\right)^3\left(2.998\times10^{8}\ \text{m/s}\right)^3}$$
$$= 1.226\times10^{17}\ \text{eV/m}^3 = 0.01964\ \text{N/m}^2\ .$$

Obviously, the radiation pressure completely dominates the baryon pressure.

(d) [2] By what factor does the pressure felt by the baryons drop from before recombination (photons + baryons) compared to after (baryons only)?

There is a subtle technicality that is that is easy to miss: just *before* recombination, the density of particles was twice as high, since there were electrons plus protons. But the baryon contribution beforehand in negligible anyway, so in round numbers

$$\frac{P_{\text{before}}}{P_{\text{after}}} = \frac{0.01964\ \text{N/m}^2}{1.336\times10^{-12}\text{N/m}^2} = 1.47\times10^{9}\ .$$

The sudden drop in pressure is why the density fluctuations, which oscillate due to pressure, suddenly transition and start to grow after recombination.

2. [10] At the time of recombination, $k_BT = 0.256\ \text{eV}$, the atoms will all be in motion.

(a) [4] According to thermodynamics, a typical thermal velocity for a non-relativistic particle is given by $E_{kin} = \frac{3}{2}k_BT$. Estimate the typical velocity of a hydrogen atom at this time.

We start by equating this energy to the formula for kinetic energy, so $E_{kin} = \frac{1}{2}mv^2$, and we find $mv^2 = 3k_BT$. Solving for the velocity, and using the mass from problem 1, we have

$$v^2 = \frac{3k_BT}{m_H} = \frac{3\left(0.256\ \text{eV}\right)\left(1.602\times10^{-19}\text{J/eV}\right)}{1.673\times10^{-27}\ \text{kg}} = 7.35\times10^{7}\ \text{m}^2/\text{s}^2\ ,$$
$$v = 8580\ \text{m/s}.$$

(b) [2] Multiply this speed by the age of the universe at this time to get an approximate distance *d* that an atom would move at the time of recombination.

From notes, the age of the universe at this time is 373,000 yr, so we have

$$d = vt = (8580\text{ m/s})(373{,}000\text{ yr})(3.156\times10^{7}\text{ s/yr}) = 1.010\times10^{17}\text{ m}.$$

(c) [2] A sphere of radius *d* found in part (b) will tend to have its density fluctuations wiped out (or at least diminished) by the atoms wandering off. Use the density of the dark matter ρ_d from problem 1b and the distance you just found to get the mass contained within this sphere. This should be the smallest structures that form, and the first structures that form. Compare to the size of a globular clusters, $10^4 - 10^7\ M_\odot$.

The volume of a sphere of radius *d* would be $\frac{4}{3}\pi d^3$, so the mass would be

$$M = V\rho_d = \tfrac{4}{3}\pi d^3\rho_d = \tfrac{4}{3}\pi\left(1.010\times10^{17}\text{ m}\right)^3\left(2.90\times10^{-18}\text{ kg/m}^3\right) = 1.250\times10^{34}\text{ kg}$$
$$= \frac{1.250\times10^{34}\text{ kg}}{1.989\times10^{30}\text{ kg}/M_\odot} = 6280\,M_\odot .$$

This is slightly lighter than the lightest globular clusters, but it's pretty close.

3. [5] At the time of matter radiation equality $z = 3400$, any fluctuations that are larger than the "horizon size" at the time will have a big disadvantage in forming structures.
(a) [2] The horizon size is simply $d = ct$. Find the horizon size at this time in kpc.

According to the notes, matter-radiation equality occurred at 57,000 yr, so we have

$$d = ct = \left(2.998\times10^{8}\text{ m/s}\right)(57{,}000\text{ yr})\left(3.156\times10^{7}\text{ s/yr}\right) = 5.393\times10^{20}\text{ m}$$
$$= \frac{5.393\times10^{20}\text{ m}}{3.086\times10^{16}\text{ m/pc}} = 1.75\times10^{4}\text{ pc} = 17.5\text{ kpc}.$$

(b) [3] Scale it up to the present. Give the answer in Mpc. Compare to the size of the Laniakea Supercluster, 160 Mpc.

Since matter-radiation equality, the universe has grown by about a factor of $z+1 = 3401$, so this scale is now

$$d_0 = d(z+1) = \left(1.75\times10^{4}\text{ pc}\right)(3401) = 5.94\times10^{7}\text{ pc} = 59.4\text{ Mpc}.$$

This is comparable to, and only slightly smaller than, the radius of the Laniakea Supercluster.

Graduate Problem: There are no graduate problems for this homework.