Physics 310/610 – Cosmology

Solution Set P

1. **[10] In class we showed that the average photon, at present, does not hit any electrons. In this problem, you will determine if the average electron is hit by a photon. The cross section is still the Thomson cross-section given in class. The density is the density of *photons*, since that's what an electron is trying to hit. The relative speed is still *c*. In the current age of the universe, how many collisions will a free electron have? Will a typical electron have been hit by at least one photon?**

The rate for collisions is given by $\Gamma = n\sigma\left(\Delta v\right)$. The cross section and relative velocity are the same, and we can get the density from the previous problem set. We have

$$\Gamma = \left(4.108\times 10^{8}\text{ m}^{-3}\right)\left(6.652\times 10^{-29}\text{ m}^{2}\right)\left(2.998\times 10^{8}\text{ m/s}\right) = 8.19\times 10^{-12}\text{ s}^{-1}$$

It is clear that this is a rare event. However, the universe is old. Multiplying by the *actual* age of the universe, we find

$$\Gamma t_0 = \left(8.19\times 10^{-12}\text{ s}^{-1}\right)\left(13.8\times 10^{9}\text{ y}\right)\left(3.156\times 10^{7}\text{ s/y}\right) = 3.57\times 10^{6}\ .$$

Obviously, each electron will have undergone many scatterings with photons over the age of the universe. In the past, when the density was higher, the number of collisions would have been many more.

2. **[10] For each of the following, estimate the thermal energy $k_B T$ of the universe. Use $g_{\text{eff}} = 3.36$.**
   (a) When primordial tritium decays (t = 17.8 y).

Since this is early in the universe (well before the matter dominated era), we use the formulas for age vs. temperature in the radiation dominated universe. We first convert the time to seconds,

$$t = \left(17.8\text{ y}\right)\left(3.156\times 10^{7}\text{ s/y}\right) = 5.62\times 10^{8}\text{ s}.$$

We now take our equation for age and solve it for the temperature:

$$t = \frac{2.42\text{ s}}{\sqrt{g_{\text{eff}}}}\left(\frac{\text{MeV}}{k_B T}\right)^2,$$

$$\left(\frac{k_B T}{\text{MeV}}\right)^2 = \frac{2.42\text{ s}}{t\sqrt{g_{\text{eff}}}},$$

$$k_B T = \left(\frac{2.42\text{ s}}{t\sqrt{g_{\text{eff}}}}\right)^{1/2}\text{ MeV}$$

Substituting in our age, we find

$$k_B T = \left(\frac{2.42 \text{ s}}{\left(5.62\times10^8 \text{ s}\right)\sqrt{3.36}} \right)^{1/2} \text{ MeV} = 4.85\times10^{-5} \text{ MeV} = 48.5 \text{ eV}$$

(b) When primordial free neutrons decay (*t* = 886 s).

We use the same formula:

$$k_B T = \left(\frac{2.42 \text{ s}}{\left(886 \text{ s}\right)\sqrt{3.36}} \right)^{1/2} \text{ MeV} = 0.0386 \text{ MeV} = 38.6 \text{ keV}$$

3. [10] For each of the following, find g_{eff}, and estimate the age of the universe in seconds.
(a) [3] At nucleosynthesis, when $k_B T = 80 \text{ keV}$.

In class, we said that for temperatures below about 100 keV, we can use $g_{\text{eff}} = 3.36$, so we have

$$t = \frac{2.42 \text{ s}}{\sqrt{g_{\text{eff}}}} \left(\frac{\text{MeV}}{k_B T} \right)^2 = \frac{2.42 \text{s}}{\sqrt{3.36}} \left(\frac{\text{MeV}}{0.080 \text{ MeV}} \right)^2 = 206 \text{ s}$$

(b) [3.5] When the thermal energy is the same as the electron rest energy, $k_B T = mc^2$. All particles are at the same temperature. In addition to photons and neutrinos, there are also electrons and positrons (*g* = 4 extra fermions).

Since all particles are at the same temperature, $g_{\text{eff}} = g_b + \frac{7}{8} g_f$. The only boson degrees of freedom are the photons with $g = 2$. There are three neutrinos plus their anti-neutrinos, which contribute $g = 6$ to the fermions. There is an additional 4 units coming from the electrons and positrons. Therefore $g_{\text{eff}} = g_b + \frac{7}{8} g_f = 2 + \frac{7}{8}\cdot 10 = 10.75$. The temperature is $k_B T = mc^2 = 0.511 \text{ MeV}$, so

$$t = \frac{2.42 \text{ s}}{\sqrt{g_{\text{eff}}}} \left(\frac{\text{MeV}}{k_B T} \right)^2 = \frac{2.42 \text{ s}}{\sqrt{10.75}} \left(\frac{\text{MeV}}{0.511 \text{ MeV}} \right)^2 = 2.83 \text{ s}$$

(c) [3.5] At the electroweak scale, $k_B T = 100 \text{ GeV}$. At this time, everything is at the same temperature, and there are *g* = 28 total spin states for bosons and *g* = 90 total spin states for fermions.

We have $g_{\text{eff}} = g_b + \frac{7}{8} g_f = 28 + \frac{7}{8}\cdot 90 = 106.75$, and therefore

$$t = \frac{2.42\text{ s}}{\sqrt{g_{\text{eff}}}}\left(\frac{\text{MeV}}{k_B T}\right)^2 = \frac{2.42\text{ s}}{\sqrt{106.75}}\left(\frac{\text{MeV}}{10^5\text{ MeV}}\right)^2 = 2.34\times10^{-11}\text{ s}$$

This is roughly the highest energy that we have direct experimental understanding of the universe, and as you can see, it takes us back to about 23 ps after the Big Bang.

Graduate Problem: Do this problem only if you are in PHY 610.

4. [15] Consider a particle moving at the speed of light in a flat universe, so *ds* = 0, where

$$ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Assume that the particle starts at *r* = 0 at time *t* = 0, and travels radially.

(a) [5] Assume first that the universe is radiation dominated, so that $a(t)\propto t^{1/2}$. Show that at time *t*, the distance the particle has traveled $d = ra(t)$ is at most $k_r ct$, and determine the pure numerical constant k_r, independent of *t*.

First we are working on a particle moving radially, so $d\theta = d\phi = 0$. Setting also $ds = 0$, we have $c^2dt^2 = a^2(t)dr^2$. Taking the positive square root, we have $dr/dt = c/a(t)$. Since it starts at $r = 0$ at $t = 0$, we therefore have

$$r(t) = \int_0^t \frac{dr(t')}{dt'}dt' = \int_0^t \frac{cdt'}{a(t')}.$$

The distance is then $d = a(t)r(t)$.

For $a(t)\propto t^{1/2}$, we write $a(t) = a_0 t^{1/2}$, and we find

$$d = a(t)r(t) = a(t)\int_0^t \frac{cdt'}{a(t')} = a_0 t^{1/2}\int_0^t \frac{cdt'}{a_0 t'^{1/2}} = ct^{1/2}\,2t'^{1/2}\Big|_0^t = 2ct.$$

So $k_r = 2$.

(b) [5] Assume second that the universe is matter dominated, so that $a(t)\propto t^{2/3}$. Show that at time *t*, the distance the particle has traveled is at most $k_m ct$, and determine the pure numerical constant k_m, independent of *t*.

For $a(t)\propto t^{2/3}$, we write $a(t) = a_0 t^{2/3}$, and we find

$$d = a(t)r(t) = a(t)\int_0^t \frac{cdt'}{a(t')} = a_0 t^{2/3}\int_0^t \frac{cdt'}{a_0 t'^{1/2}} = ct^{2/3}\,3t'^{1/3}\Big|_0^t = 3ct.$$

So $k_m = 3$.

(c) [5] Assume third that the universe is cosmological constant dominated, so that $a(t) \propto \exp(H_1 t)$. Show that in this case, for sufficient time, the distance traveled is greater than any multiple of *ct*.

For $a(t) \propto \exp(H_1 t)$, we write $a(t) = a_0 \exp(H_1 t)$, and we find

$$d = a(t) r(t) = a(t) \int_0^t \frac{c\,dt'}{a(t')} = a_0 \exp(H_\Lambda t) \int_0^t \frac{c\,dt'}{a_0 \exp(H_\Lambda t')} = c \exp(H_1 t) \left[-H_1^{-1} \exp(-H_1 t) \right]_0^t$$

$$= \frac{c}{H_1} \left[-1 + \exp(H_1 t) \right].$$

Since exponentials beat all power laws, this at large times will beat all multiples of *ct*.