Physics 310/610 – Cosmology

Solution Set K

1. **[10] A distant star is lensed by the Sun. The Sun is at d_L = 1.00 AU, and the star is so distant that effectively $d_S \approx d_{LS}$**

   (a) [5] Find the Einstein radius θ_E in arc-seconds for distant stars lensed by the Sun.

   The Einstein radius is given by

$$\theta_E^2 = \frac{4GMd_{LS}}{c^2 d_L d_S} = \frac{4GM_\odot}{c^2 d_L} = \frac{4\left(6.674\times10^{-11}\ \text{kg}^{-1}\text{m}^3\text{s}^{-2}\right)\left(1.989\times10^{30}\ \text{kg}\right)}{\left(2.998\times10^{8}\ \text{m/s}\right)^2 (1\ \text{AU})\left(1.496\times10^{11}\ \text{m/AU}\right)} = 3.949\times10^{-8}\ ,$$

$$\theta_E = \sqrt{3.949\times10^{-8}}\,\text{rad}\cdot\left(\frac{180^\circ}{\pi\ \text{rad}}\right)\left(\frac{60'}{1^\circ}\right)\left(\frac{60''}{1'}\right) = 40.99''.$$

   (b) [5] A star is positioned such that it would normally be just at the edge of the Sun, $\beta = 30' = 1800''$. There will be two images, $\theta_\pm$. Show that the inner one is invisible (because it is less than the Sun's radius away), and find or approximate how much the outer one is displaced, $\theta_+ - \beta$, in arc-seconds.

   We simply use the formula

$$\theta_\pm = \tfrac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right) = \tfrac{1}{2}\left(1800 \pm \sqrt{1800^2 + 4\cdot(40.99)^2}\right)''$$

$$= \tfrac{1}{2}\left(1800 \pm 1801.87\right)'' = \begin{cases} 1800.93'' & \text{for } + \\ -0.93'' & \text{for } - \end{cases}$$

Obviously, the second one is way inside the Sun. The outer one is shifted outwards by 0.93″.

2. **[10] Suppose a lensing object has mass $M = 1.00M_\odot$ is exactly half way to the source, so $d_S = 2d_L$. Show that the Einstein radius θ_E as a function of the distance d_L take the form $\theta_E = C\sqrt{\text{kpc}/d_L}$, and determine the constant *C* in milli-arc seconds. It is very difficult to measure variations in direction at the milli-arc second scale.**

We simply use the formula

$$\theta_E^2 = \frac{4GMd_{LS}}{c^2 d_L d_S} = \frac{2GM_\odot}{c^2 d_L} = \frac{2\left(6.674\times10^{-11}\ \text{kg}^{-1}\text{m}^3\text{s}^{-2}\right)\left(1.989\times10^{30}\ \text{kg}\right)}{\left(2.998\times10^8\ \text{m/s}\right)^2(\text{kpc})}\left(\frac{\text{kpc}}{d_L}\right)$$

$$= \frac{2954\ \text{m}}{10^3\times3.086\times10^{16}\ \text{m}}\left(\frac{\text{kpc}}{d_L}\right) = 9.572\times10^{-17}\left(\frac{\text{kpc}}{d_L}\right),$$

$$\theta_E = \sqrt{9.572\times10^{-17}}\sqrt{\frac{\text{kpc}}{d_L}}\ \text{rad}\cdot\left(\frac{180^\circ}{\pi\ \text{rad}}\right)\left(\frac{60'}{1^\circ}\right)\left(\frac{60''}{1'}\right)\left(\frac{1000\ \text{mas}}{1''}\right) = (2.02\ \text{mas})\sqrt{\frac{\text{kpc}}{d_L}}.$$

Since angular deflections below a mas are virtually impossible to see, this deflection is effectively invisible.

3. **[10] Show that the amplification of a star by gravitational microlensing is an amplification; that is, show**

$$A_{tot} = \frac{1}{2}\left(\frac{\beta}{\sqrt{\beta^2+4\theta_E^2}} + \frac{\sqrt{\beta^2+4\theta_E^2}}{\beta}\right) > 1.$$

We multiply both sides of this equation by $\beta\sqrt{\beta^2+4\theta_E^2}$ (a positive number) to yield

$$\tfrac{1}{2}\left[\beta^2 + \left(\beta^2+4\theta_E^2\right)\right] > \beta\sqrt{\beta^2+4\theta_E^2}$$

We now square both sides to yield

$$\left(\beta^2+2\theta_E^2\right)^2 > \left(\beta\sqrt{\beta^2+4\theta_E^2}\right)^2,$$

$$\beta^4 + 4\beta^2\theta_E^2 + 4\theta_E^4 > \beta^2\left(\beta^2+4\theta_E^2\right) = \beta^4 + 4\beta^2\theta_E^2.$$

Cancelling the common terms, this is equivalent to $4\theta_E^4 > 0$, an obviously true statement.

Physics 610: Only do this problem if you are in the graduate version of this course

4. [10] The amplification is always greater than one. So why do we have to get things lined up pretty well before we get lensing events? Taylor expand the formula for amplification in powers of θ_E/β to at least fourth order in θ_E. Based on your Taylor expansion, estimate the largest angle β you can miss by to have even a 0.1% amplification.

We can do this by hand or with the help of the binomial theorem. We first rewrite the amplification in the form

$$A_{tot} = \tfrac{1}{2}\left[\left(1+4\theta_E^2/\beta^2\right)^{-1/2} + \left(1+4\theta_E^2/\beta^2\right)^{1/2}\right]$$

The binomial expansion tells us $(1+\varepsilon)^n = 1 + n\varepsilon + \tfrac{1}{2}n(n-1)\varepsilon^2 + \tfrac{1}{6}n(n-1)(n-2)\varepsilon^3 + \cdots$. Applying this, we have

$$A_{tot} \approx \frac{1}{2}\left[1+\left(-\frac{1}{2}\right)\frac{4\theta_E^2}{\beta^2}+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{4\theta_E^2}{\beta^2}\right)^2+1+\left(\frac{1}{2}\right)\frac{4\theta_E^2}{\beta^2}+\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{4\theta_E^2}{\beta^2}\right)^2\right]$$

$$=\frac{1}{2}\left[2+\left(-\frac{1}{2}+\frac{1}{2}\right)\frac{4\theta_E^2}{\beta^2}+\left(\frac{3}{8}-\frac{1}{8}\right)\left(\frac{4\theta_E^2}{\beta^2}\right)^2\right]=1+\frac{2\theta_E^4}{\beta^4}.$$

Setting $2\theta_E^4/\beta^4 = 10^{-3}$, we find $\beta = \left(2000\theta_E^4\right)^{1/4} = 6.7\theta_E$. So even such a tiny amplification requires that you be closer than seven Einstein radii away.

If you want to keep more terms, you can easily let Maple do it for you:

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> assume(beta>0);series(1/2*(beta/sqrt(beta^2+4*theta^2)+
  sqrt(beta^2+4*theta^2)/beta),theta,11);
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You can also do all the terms by hand, with a bit of work.

$$A_{tot} = 1+\sum_{n=2}^{\infty}(-1)^n\frac{2(2n-2)!}{n!(n-2)!}\frac{\theta_E^{2n}}{\beta^{2n}} = 1+\frac{2\theta_E^4}{\beta^4}-\frac{8\theta_E^6}{\beta^6}+\frac{30\theta_E^8}{\beta^8}-\frac{112\theta_E^{10}}{\beta^{10}}+\cdots.$$