Physics 310/610 – Cosmology

Solution Set H

1. [25] In class we modeled both the thin disk and thick disk as stars with number density given by

$$n = n_0 e^{-R/h_r} e^{-|z|/h_z}$$

For the thin disk, assume $h_r = 3500\text{ pc}$ **and** $h_z = 350\text{ pc}$**. For the thick disk,** $h_r = 3500\text{ pc}$ **and** $h_z = 1200\text{ pc}$**.**

(a) [5] Find a formula for the total number of stars in terms of *n*0, *hr*, and *hz*. Make sure you are working in cylindrical coordinates!

To find this, we simply integrate the number density over cylindrical coordinates, so we have

$$\begin{aligned} N &= \int_0^\infty R\,dR \int_{-\infty}^\infty dz \int_0^{2\pi} d\phi\, n(R,z) = 2\pi n_0 \int_0^\infty e^{-R/h_r} R\,dR \int_{-\infty}^\infty e^{-|z|/h_z} dz \\ &= 2\pi n_0 \left[-Rh_r e^{-R/h_r} - h_r^2 e^{-R/h_r} \right]_0^\infty 2\int_0^\infty e^{-z/h_z} dz = 2\pi n_0 \left[0 + h_r^2 \right] 2 \left[-h_z e^{-z/h_z} \right]_0^\infty = 4\pi n_0 h_r^2 h_z\,. \end{aligned}$$

(b) [6] What fraction of the stars are at a radius *R* < *R*0 for any given *R*0? Find the fraction of the disk stars closer than the Sun at $R_0 = 8300\text{ pc}$**.**

We simply replace the upper limit on the radial integral by R_0, so we find

$$\begin{aligned} N(R<R_0) &= \int_0^{R_0} R\,dR \int_{-\infty}^\infty dz \int_0^{2\pi} d\phi\, n(R,z) = 2\pi n_0 \left[-Rh_r e^{-R/h_r} - h_r^2 e^{-R/h_r} \right]_0^{R_0} 2\int_0^\infty e^{-z/h_z} dz \\ &= 4\pi n_0 \left[-R_0 h_r e^{-R_0/h_r} - h_r^2 e^{-R_0/h_r} + h_r^2 \right] h_z \end{aligned}$$

The fraction closer than the Sun is then

$$\frac{N(R<R_0)}{N} = \frac{4\pi n_0 \left[-R_0 h_r e^{-R_0/h_r} - h_r^2 e^{-R_0/h_r} + h_r^2 \right] h_z}{4\pi n_0 h_r^2 h_z} = 1 - \left(1 + \frac{R_0}{h_r}\right) e^{-R_0/h_r}\,.$$

Substituting numbers, we then have

$$\frac{N(R<R_0)}{N} = 1 - \left(1 + \frac{8300}{3500}\right) e^{-8300/3500} = 1 - (1 + 2.371) e^{-2.371} = 0.685 = 68.5\%.$$

(c) [7] The Sun is about at $z = 0$**, and the local density of thin disk stars is about 0.14 pc^{-3}. Find the central density *n*0, and the total number of stars in the thin disk. Multiply by 0.6 $M_\odot$ to get the approximate mass of the thin disk.**

By equating the formula for the density to the value at the Sun, we see that

$$0.14\ \text{pc}^{-3} = n_0 e^{-R_0/h_r} e^{-|z_0|/h_z} = n_0 e^{-2.371} ,$$
$$n_0 = e^{2.371}\left(0.14\ \text{pc}^{-3}\right) = 1.50\ \text{pc}^{-3} .$$

We then substitute this into the formula for the total number of stars to get

$$N = 4\pi n_0 h_r^2 h_z = 4\pi\left(1.499\ \text{pc}^{-3}\right)\left(3500\ \text{pc}\right)^2\left(350\,\text{pc}\right) = 8.08\times10^{10} .$$

Multiplying by 0.6 $M_\odot$, a typical star mass, we get a total mass of $4.85\times10^{10}\ M_\odot$. The actual disk is a bit heavier than this; I don't know why the answer came out wrong.

(d) [7] The local density of thick disk stars is about 0.002 pc^{-3}. Repeat part (b) for the thick disk. What fraction of the disk stars are thick disk stars?

The computation is identical with the previous part. We have

$$n_0 = e^{2.371}\left(0.002\ \text{pc}^{-3}\right) = 0.0214\ \text{pc}^{-3} ,$$
$$N = 4\pi n_0 h_r^2 h_z = 4\pi\left(0.0214\ \text{pc}^{-3}\right)\left(3500\ \text{pc}\right)^2\left(1200\,\text{pc}\right) = 3.96\times10^{9} .$$

We then find the mass as before, which is $2.37\times10^{9}\ M_\odot$. The fraction which are thick disk are

$$\frac{N_{\text{thick}}}{N_{tot}} = \frac{3.96\times10^{8}}{8.08\times10^{10}+3.96\times10^{8}} = 0.0467 = 4.7\%.$$

So roughly five percent of disk stars are thick disk stars, even though the fraction of stars near us is much lower than this.

2. [5] Find the Schwarzschild radius for a black hole with mass equal to the mass of the Earth (in cm), for the Sun (in km) and for the black hole at the center of our galaxy (in AU) assuming a mass of $M = 4.3\times10^{6}\ M_\odot$. In the case of the Earth, draw a circle of approximately the correct radius for such a black hole on your paper. If it is too big to fit on your paper, or if it is too small to see, you have made an error.

The formula for the radius of a black hole is simply $R_s = 2GM/c^2$, which works out to

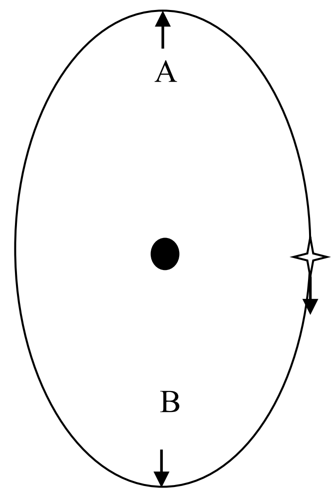
$$R_{s\oplus} = \frac{2\left(6.674\times10^{-11}\ \text{m}^3/\text{kg/s}^2\right)\left(5.972\times10^{24}\ \text{kg}\right)}{\left(2.998\times10^{8}\ \text{m/s}\right)^2} = 0.008869\ \text{m} = 0.8869\ \text{cm},$$

$$R_{s\odot} = \frac{2\left(6.674\times10^{-11}\ \text{m}^3/\text{kg/s}^2\right)\left(1.989\times10^{30}\ \text{kg}\right)}{\left(2.998\times10^{8}\ \text{m/s}\right)^2} = 2954\ \text{m} = 2.954\ \text{km}$$

$$R_{s,bh} = R_{s\odot}\frac{M_{bh}}{M_\odot} = \frac{\left(2954\ \text{m}\right)\left(4.3\times10^{6}\right)}{1.496\times10^{11}\ \text{m/AU}} = 0.085\ \text{AU}.$$

The diameter of the Earth-mass black hole would be 1.77 cm, which is about the size of a dime.

PHY 610 – Do the following problem only if you are in PHY 610



3. **[30] An alien race with a similar situation to ours suspects there is a black hole at the center of their galaxy. They observe that it is circled by a star which they presume is moving in a perfectly circular orbit. The orbit does not *look* circular because the orbit it tilted with respect to the angle of observation. At right is the orbit as measured, with 1 cm representing 10 mas.**
   (a) **[4] Based on the shape of the observed orbit, what is the tilt of the orbit α compared to the normal? The angle is defined so that if $\alpha = 0°$ it would look like a perfect circle, and if $\alpha = 90°$ it would look like a flat line.**

The vertical extent of this ellipse shows the true angular diameter of the orbit. The horizontal is shrunk by a factor of $\cos\alpha$. The ratio of the two diameters tells us $\cos\alpha$. Vertically, I measure it to be about 6.30 cm tall. Horizontally I measure it to be about 3.91 cm wide. I therefore have

$$\cos\alpha = 3.91/6.30 = 0.621, \quad \text{so} \quad \alpha = \cos^{-1}(0.621) = 51.6°.$$

   (b) **[8] A spectral line that would normally be at 589.00 nm is discovered to have a wavelength of 592.29 nm at point A and 585.73 at point B. At the position drawn, it is very close to the normal wavelength. What is the approximate speed of the star around the black hole?**

According to the non-relativistic approximation, $\lambda_0/\lambda = 1 + z = 1 + v_r/c$. We can therefore get the radial speed in each of the two positions:

$$v_{rA} = c\left(\frac{\lambda_{A0}}{\lambda} - 1\right) = \left(2.998\times10^5 \text{ km/s}\right)\left(\frac{592.29 \text{ nm}}{589.00 \text{ nm}} - 1\right) = 1675 \text{ km/s},$$
$$v_{rB} = c\left(\frac{\lambda_{B0}}{\lambda} - 1\right) = \left(2.998\times10^5 \text{ km/s}\right)\left(\frac{585.73 \text{ nm}}{589.00 \text{ nm}} - 1\right) = -1664 \text{ km/s}.$$

The fact that it is doing neither at the point marked indicates that it is moving neither towards nor away from the aliens, which really means that the alien star is neither moving towards nor away from the center of the galaxy. For a circular orbit, the radial velocities should be equal, and to the number of significant figures given, they are essentially equal. However, we are seeing only the *radial* motion, and for this angle, this is given by the true speed multiplied by $\sin\alpha$. In other words, the true speed is

$$v_r = v\sin\alpha, \quad \text{so} \quad v = \frac{v_r}{\sin\alpha} = \frac{1670 \text{ km/s}}{\sin(51.6°)} = 2130 \text{ km/s}$$

(c) [6] The star completes an entire orbit in 8.10 Earth years. What is the radius of the orbit, in AU?

The circumference of the orbit is simply speed times time. The radius is this divided by 2π, so

$$R = \frac{C}{2\pi} = \frac{vT}{2\pi} = \frac{(2130\,\text{km/s})(8.10\text{ y})(3.156\times10^{7}\text{ s/y})}{2\pi(1.496\times10^{8}\text{ km/AU})} = 579\text{ AU}.$$

(d) [6] What is the mass of the black hole, in solar masses?

We equate the centripetal acceleration $a = v^2/R$ to the gravitational acceleration $g = GM/R^2$, and find $GM = v^2 R$. Solving for the mass, we have

$$M = \frac{v^2 R}{G} = \frac{(2.130\times10^{6}\text{ m/s})^2(579\text{ AU})(1.496\times10^{11}\text{ m/AU})}{(6.674\times10^{-11}\text{ m}^3/\text{kg/s}^2)(1.989\times10^{30}\text{ kg}/M_\odot)} = 2.96\times10^{6}\ M_\odot.$$

It came out a little lighter than our black hole.

(e) [6] What is the distance of the black hole?

The actual angular radius is 579 AU, but its apparent radius is 31.5 mas. The distance is then given by

$$d = \frac{s}{\theta} = \frac{579\text{ AU}}{0.0315''}\cdot\frac{1\text{ pc}\cdot1''}{1\text{ AU}\cdot1\text{ rad}} = 18{,}400\text{ pc} = 18.4\text{ kpc}$$

This came out a little large, but so what? Apparently, they live in an especially large galaxy.