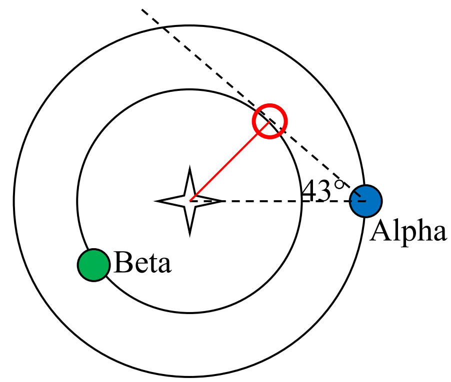
Physics 310 – Cosmology

Solution Set E



1. **[12] The occupants of planet Alpha have determined that both they, and another planet Beta orbit their star in circular orbits, as sketched at right. Over the course of the year, they have discovered that planet Beta is never more than 43.0 degrees from their star.**
   (a) [2] What angle of the triangle Alpha-Beta-Star is a right angle when Beta is in the right position to be at this maximum angle?

I have drawn in the appropriate position of Beta and filled in the third side of the triangle in red. It should be clear that the Beta angle is ninety degrees. This is a theorem of geometry: a tangent to a circle is perpendicular to the radius drawn to that point.

(b) [3] When Beta is at this maximum angle, a radar signal is sent to Beta which returns exactly 782 seconds later. How far is Beta from Alpha at this moment, in AU?

If it took 782 seconds to make the round trip, it must have taken half of this, or 391 seconds to go one way. Multiplying by the speed of light give the distance

$$d = ct = \left(2.998\times10^{8}\text{ m/s}\right)\left(391\text{ s}\right)\frac{1\text{ AU}}{1.496\times10^{11}\text{ m}} = 0.7835\text{ AU}$$

(c) [4] What is the distance of Alpha from the star, and of Beta from the star, in AU?

Since it's a right triangle, we can use relations for right triangles to get these two distances:

$$a_\alpha = \frac{d}{\cos(43°)} = \frac{0.7835\text{ AU}}{\cos(43°)} = 1.074\text{ AU},$$
$$a_\beta = d\tan(43°) = (0.7835\text{ AU})\tan(43°) = 0.731\text{ AU}$$

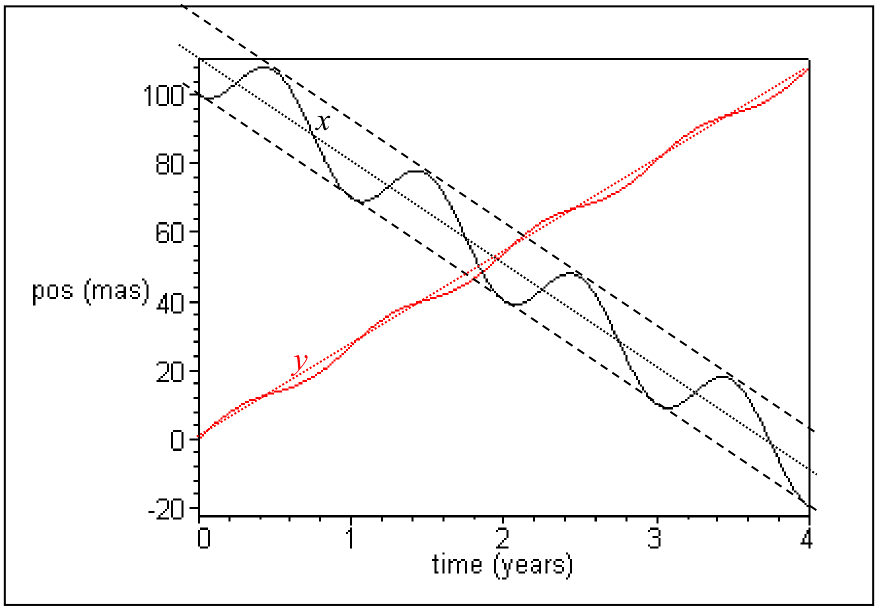
(d) [3] Observations indicate that at this same point, Beta has an angular *diameter* of 37.0″ (37 arc-seconds). What is the *radius* of the planet Beta, in km?

We can use the small angle formula to get the diameter. However, this can only be done if we work in radians. So we need

$$s = \theta d = (37'')(0.7835\text{ AU})\frac{1°}{3600''}\frac{2\pi\text{ rad}}{360°}\frac{1.496\times10^{8}\text{ km}}{1\text{ AU}} = 21{,}000\text{ km}$$

This is the diameter, so the radius is half this, or about 10,500 km.

2. **[18] A star at β = 12° above the ecliptic has its position accurately marked in the sky over a four year period. The position as a function of time ends up looking like the accompanying graphs.**
   (a) [4] Find the angular velocity of proper motion in the *x*- and *y*-directions



I have added lines showing the rate of growth of the two curves (ignoring the wiggles) to help guide the eye. The *x*-curve seems to drop about 120 mas in four years, for an angular speed of $\mu_x = -30$ mas/y. In the *y*-direction, we see that it is rising approximately 108 mas in the same four years, or $\mu_y = +27$ mas/y. So far so good!

(b) [4] Find the parallax in mas, and the distance to the star in pc. The *x*-direction corresponds to the major axis of the parallax ellipse.

The parallax is the amount the *x*-curve varies up and down from its central value. By eyeballing, I'd say the two limit curves represented by the dashed curves are about 10 mas up and down from the median line drawn. Hence the parallax is estimated as 10 mas, which is the same as 0.010″.

To get the distance in parsecs, we'd then simply take the reciprocal of this number,

$$d = \frac{1}{p} = \frac{1}{0.010} = 100 \text{ pc},$$

This answer is probably good to about ten percent.

(c) [4] Find the transverse velocity components v_x and v_y.

We simply use the formula $v = \mu d$, but we must convert the angular velocity into radians, as well as various other minor complications. We find

$$v_x = \mu_x d = \left(-0.030''/\text{y}\right)\left(100\text{ pc}\right)\frac{1\text{ AU}\cdot\text{rad}}{1\text{ pc}\cdot 1''}\frac{1.496\times 10^8\text{ km}}{1\text{ AU}}\frac{1\text{ y}}{3.156\times 10^7\text{ s}} = -14\text{ km/s},$$

$$v_y = \mu_y d = \left(0.027''/\text{y}\right)\left(100\text{ pc}\right)\frac{1\text{ AU}\cdot\text{rad}}{1\text{ pc}\cdot 1''}\frac{1.496\times 10^8\text{ km}}{1\text{ AU}}\frac{1\text{ y}}{3.156\times 10^7\text{ s}} = 13\text{ km/s}.$$

(d) [4] The spectrum of the star is measured, and it is found that a spectral line that should be at 710.43 nm is actually at 710.61 nm. What is the radial velocity of the star, and is it towards us or away from us?

The wavelength shift is certainly small enough that we can use the non-relativistic limit, $z = v_r/c$. Recalling that $\lambda_0/\lambda = 1+z$, we see that

$$\frac{v}{c} = z = \frac{\lambda_0}{\lambda} - 1 = \frac{710.61}{710.43} - 1 = .00025,$$
$$v = 0.00025\left(2.998\times 10^5\text{ km/s}\right) = 76\text{ km/s}$$

This rather large radial velocity shows that your professor was too lazy to work out realistic numbers when he made up the problem.

(e) [2] What is the total speed of the star relative to the solar system?

This is straightforward:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\left(-14\text{ km/s}\right)^2 + \left(13\text{ km/s}\right)^2 + \left(76\text{ km/s}\right)^2} = 78\text{ km/s}.$$

Graduate Problems: Only do this problem if you are in PHY 610

3. **[15] A star at distance *d* is moving directly away from us at velocity *v*, which is related to the red-shift *z* as described in class. To simplify things, assume that the star has radius *R* and a uniform temperature *T* as viewed in its own frame**
   (a) **Because of Lorentz contraction, the star *as viewed by us* will no longer be a sphere. Can we nonetheless use the formula $\alpha = 2R/d$ to get the angular size as viewed by us, or would it have to be modified?**

This is pretty trivial. The object Lorentz contracts only in the longitudinal direction, so the transverse size is still $2R$.

   (b) **The expected number of photons occupying any particular photon state at temperature *T* is given by**

$$n(E) = \frac{1}{e^{E/k_B T} - 1}$$

For a photon moving towards us, what would be the energy *E′* that we observe, in terms of *E* and the red-shift *z*? Show that the distribution is still thermal, but with different temperature *T′* as observed by us.

A massless photon moving towards us with energy E would also have momentum $p = -E/c$. Now, the object is moving away from us at velocity v, but we can simply treat that as if we were moving backward by velocity v. The energy as viewed by us would then be given by

$$E' = \gamma\left[E - (-v)p\right] = \gamma\left(E - vE/c\right) = \frac{1 - v/c}{\sqrt{1 - v^2/c^2}}E = \sqrt{\frac{1 - v/c}{1 + v/c}}E = \frac{E}{1+z}.$$

It follows that $E = (1+z)E'$. Since a state that is occupied will still be occupied, the occupation number can be written in terms of E' as

$$n'(E') = n(E) = \frac{1}{e^{E/k_B T} - 1} = \frac{1}{e^{E'(1+z)/k_B T} - 1},$$

$$n'(E') = \frac{1}{e^{E'/k_B T'} - 1} \quad \text{where} \quad T' = \frac{T}{1+z}.$$

Hence a thermal distribution will still look thermal, but with its temperature reduced by a factor of $1 + z$.

   (c) **Argue that the apparent brightness of a star moving away from us will simply be reduced by a factor of $(1+z)^4$ compared to a star that is not moving away from us.**

The distribution still looks thermal, but the apparent temperature has been reduced by a factor of $1 + z$. The apparent amount of light headed our way will then still be given by $\sigma T'^4 = \sigma T^4 (1+z)^{-4}$. Since the angular size of the star is still the same, and the distance is still the same, this just implies an overall diminution of the brightness by a factor of $(1+z)^4$.