Name ___________________

Solutions to Final Exam
December 14, 2017

This test consists of five parts. In parts II through V, PHY 310 students can skip one question of those offered, while PHY 610 students must answer all questions.

Part I: Multiple Choice (mixed new and review questions) [50 points] (2 points each)
PHY 310/610: For each question, choose the best answer

1. At the end of neutron/proton freezeout, how did the number of neutrons compare to the number of protons?
   A) It was almost all neutrons
   B) There were several times more neutrons than protons (about 6:1)
   C) There were almost equal numbers of protons and neutrons
   D) There were several times more protons than neutrons (about 6:1)
   E) It was almost all protons

2. Which of the following is closest to the current age of the universe?
   A) 11.8 Gyr B) 12.8 Gyr **C) 13.8 Gyr** D) 14.8 Gyr E) 15.8 Gyr

3. Which of the following explains why Hubble's Law becomes poor at large red-shift
   A) The distances are so great you are looking at things in the past (only)
   B) There are different kinds of distances (luminosity distance? Current distance?) (only)
   C) The curvature of spacetime makes it measuring distances complicated (only)
   D) All of the above are correct
   E) None of the above are correct

4. Water freezes at 273 K. What was the approximate value of the red-shift *z* when the universe was at this temperature?
   A) 99 B) 9 C) 0 D) –0.99 E) –0.9

5. The most likely ultimate cause of structure in the current universe arose when?
   A) From black holes introducing non-uniformity
   B) From random photons pushing matter together at recombination
   C) From coalescence around very massive dark matter particles
   D) From density fluctuations at the time of quark confinement
   E) From quantum mechanical fluctuations that got made very large during inflation

6. The reason Cepheid Variable stars are useful for measuring distances is that
   A) We can bounce radio waves off of them and measure the timing for the signal to return
   B) These stars have consistent size, so we can compare the angular size to the actual size
   C) These stars are normally all at the same distance, so you automatically know the distance
   D) These stars have a relationship between their period and luminosity
   E) These stars are always close enough to measure their distance via parallax

7. Which of the following was <u>not</u> a possibility that was discussed for how the universe made an excess of baryons over anti-baryons?
   A) Decay of particles at the grand unified theory (GUT) scale
   B) Production at the electroweak phase transition via sphalerons
   C) Production by black holes in the early universe
   D) Production with the help of supersymmetry at the electroweak scale via sphalerons
   E) Neutrino-driven baryogenesis, with a neutrino asymmetry converted into a baryon asymmetry

8. How is it that black holes are believed to be able to evaporate in the future universe?
   A) Over time, black holes turn into white holes, which only allow stuff to escape from them
   B) Quantum mechanics says that particle/anti-particles can appear, and one can get eaten by the black hole while the other escapes
   C) They split into pairs which eventually become so small they are invisible
   D) The matter that falls into them goes through a wormhole and escapes elsewhere
   E) They don't really evaporate, they just are invisible because they are black

9. From our standpoint, all objects in the universe seem to be rushing away from us with a velocity proportional to their distance. What would be observed from a distant galaxy?
   A) All objects would be rushing past them in one direction
   B) All objects would be rushing away in some directions and rushing towards them in others
   C) All objects would be rushing away, but not in a way proportional to distance
   D) All objects would be rushing away, proportional to distance, but with a different proportionality (different Hubble's constant)
   E) The same thing would be observed

10. Our galaxy's classification is probably approximately
    A) E0 B) E7 C) SAd D) SAm **E) SBb**

11. The nearest large galaxy besides our own is called
    A) Andromeda B) Virgo C) Coma D) Fornax E) Milky Way

12. Which of the following is the largest contributor to the current mass density of the universe?
    A) Dark matter B) Baryons **C) Dark energy** D) Neutrinos E) Radiation

13. Which of the following was discussed as a possibility for the creation of the universe
    A) Eternal/chaotic inflation, where most of the universe is forever inflating but certain small regions escaped
    B) Creation from a white hole, sort of a reverse-time black hole where things can only come out
    C) A primordial thick soup of particles and anti-particles annihilated to create an explosion
    D) As a giant sneeze from the Great Green Arkleseizure
    E) None of these were discussed

14. Which kind of galaxies typically have lots of gas, dust, and star formation?
    A) Ellipticals (only)
    B) Spirals (only)
    C) Barred spirals (only)
    D) Spirals and barred spirals, but not ellipticals
    E) Spirals, barred spirals, and ellipticals

15. In the future, our best guess about what will happen to the universe is it will
    A) Expand for a while, then recontract to a point in the Big Crunch
    B) Expand, but at an ever slowing rate, eventually coasting at a constant speed
    C) Expand exponentially forever, so that things end up very far apart
    D) Expand to a fixed constant size, and remain that size forever
    E) None of the above

16. The bottom quark has a mass of about $mc^2 = 4.2$ GeV. What was the approximate temperature $k_B T$ when bottom quarks annihilated and disappeared?
    A) 0.7 GeV **B) 1.4 GeV** C) 4.2 GeV D) 12.6 GeV E) 25.2 GeV

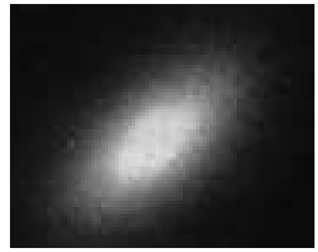
17. How do typical stars in the disk of a spiral galaxy orbit?
    A) Approximately circular orbits in the plane of the disk
    B) Highly eccentric orbits, but in the plane of the disk
    C) Approximately circular orbits, but not in the plane of the disk
    D) Highly eccentric orbits not in the plane of the disk
    E) None of the above

18. Which of the following coincidences were not discussed as being critical to the universe being conducive to life and intelligence?
    A) The value of $\Omega = 1$
    B) The vacuum energy density is very low
    C) The near perfect resonance of the ^{8}Be nucleus and an excited state of ^{12}C
    D) The perfect muon mass to allow muons to increase the temperature just right
    E) In fact, all of these were discussed

19. Which of the following is not part of our address in the Universe?
    A) Milky Way Galaxy
    B) Laniakea Supercluster
    C) Virgo Cluster
    D) Local Group
    E) Actually, we are a member of all of these

20. The cosmic microwave background radiation shows us the universe at the time of
    A) First structure B) Nucelosynthesis C) Inflation D) Baryogenesis **E) Recombination**

21. Which of the following combinations of quarks and anti-quarks do not occur in our universe?
   A) Three quarks
   B) Two quarks
   C) Three anti-quarks
   D) One quark and one anti-quark
   E) Actually, all of these occur in nature

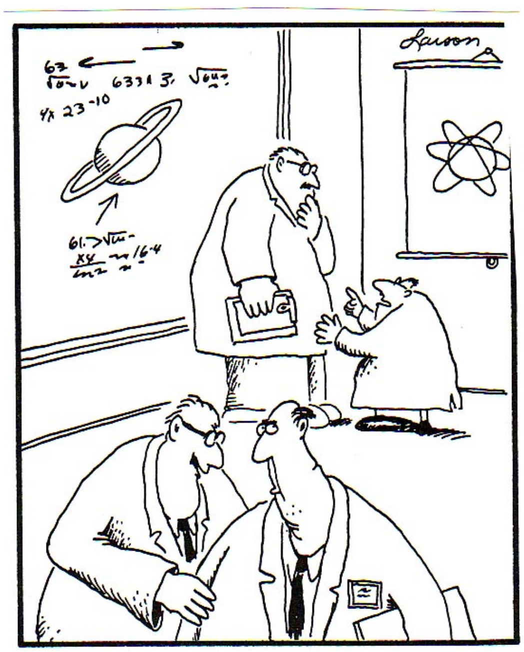


22. The classification of the galaxy at right is probably about
   A) E0 **B) E7** C) SAd D) SAm E) SBb

23. How do we know that the dark matter does not consist of white dwarfs, neutron stars, black holes, planets, or other similar massive compact halo objects (MACHOs)?
   A) Many of these would be easily visible with modern telescopes
   B) These objects have so much gravity we would observe them distorting the solar system
   C) The accretion disks around any of these objects would produce X-rays that could be observed
   D) These objects would cause gravitational microlensing of bright stars behind them
   E) Since most of these would have been stars in the past, we should see them when we look at very distant galaxies

24. Which of the following events occurred earliest?
   A) Neutrino decoupling
   B) Electron-positron annihilation
   C) Nucleosynthesis
   D) Neutron/proton ratio freezeout
   E) Matter-radiation equality

25. Which of the following is not explained in terms of inflation
   A) Why the universe has Ω extremely close to one
   B) Why the universe is seems to have relatively constant composition
   C) Why the cosmic microwave background radiation (CMBR) temperature is so uniform
   D) Where the small fluctuations in the CMBR come from
   E) Why there are more electrons than positrons in the universe

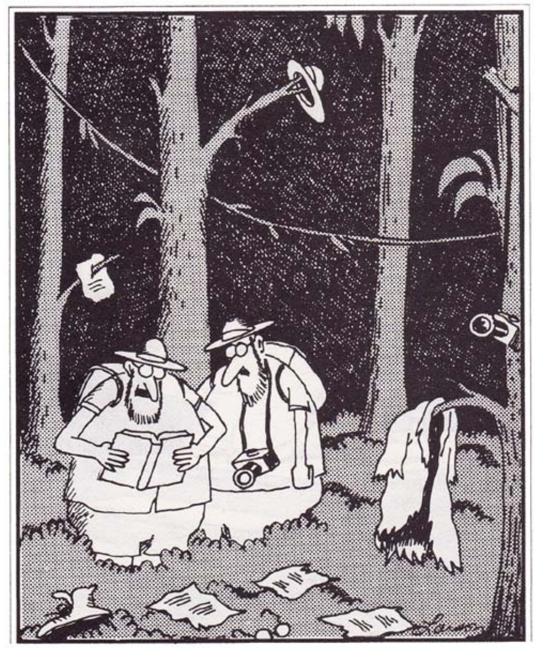


"There goes Williams again . . . trying to win support for his Little Bang theory."

Part II: Short answer (review material) [20 points/30 points] (10 points each)
PHY 310: Choose **two** of the following three questions and give a short answer (1-3 sentences)
PHY 610: Answer all three questions

26. Explain how we know that the dark matter in galaxies is not concentrated at the center.

We can measure the rotation rate in nearby galaxies (as a function of distance from the center) by using the Doppler shift of various spectral lines. If the mass were concentrated in the center, then the velocity curve should fall off as the square root of the distance, the prediction of Newton's laws. But it actually is roughly flat, which is only possible if the mass is spread out over a larger region.



"Here's the last entry in Carlson's journal: 'Having won their confidence, tomorrow I shall test the humor of these giant but gentle primates with a simple joy-buzzer handshake.'"

27. It is hypothesized that quasars and blazars may, in fact, be similar or even identical objectss. How is this possible, given that they have significantly different properties? You may have to draw a sketch to explain this.

All galaxies can be viewed only from one angle, because they are too far away to view from multiple angles. Many quasars have jets shooting out of them. If you happen to be directly in line with the jet, you would see a blazar, since you are looking straight down the barrel of the jet coming out, and it beams its light preferentially forwards. If you are at an angle (which is more common), you would mostly see the hot gas near the center, not the light from the relativistic jet. Hence the same object can look different from different directions.

28. Explain qualitatively the importance of the "turn-off point" for estimating the age of a cluster of stars.

Main sequence stars fall within a diagonal band on the Hertzsprung-Russell diagram, with the highest mass stars in the top left (luminous and hot) and the low mass stars in the lower right (dim and cool). As the cluster ages, the highest mass stars move off of the main sequence and into the red giant (or other) stages. By seeing the point at which stars have "turned off" of the main sequence (the turn-off point), you can tell which stars have finished their main sequence, and hence can tell how old the cluster is.

Part III: Short answer (new material) [30 points/40 points] (10 points each)
PHY 310: Choose **three** of the following four questions and give a short answer (1-3 sentences)
PHY 610: Answer all four questions

29. There is strong evidence that there are more baryons than anti-baryons in the universe. Give at least three criteria that any process that produces this asymmetry must have

Any process that produces baryons must satisfy three criteria: (1) it must violate baryon number, (2) it must be out of thermal equilibrium, and (3) it must violate both C and CP symmetry. C is the symmetry that says particles act the same as anti-particles, and CP is that combined with mirror image, or parity symmetry.

30. Give at least two reasons why it is unlikely that neutrinos are the dark matter

Neutrinos have very low mass, and therefore if they were the dark matter, at their temperature they would have high velocity. This causes them to free stream, moving from high to low density, and wiping out any density fluctuations which could result in galaxies, clusters, and other structures.

Neutrinos, also being fermions, must obey the Pauli exclusion principle, and fitting that many low-mass fermions into a galaxy would cause them to have such high velocity that they would, in fact, escape from the galaxy.

31. It is believed that the temperature of the neutrinos is a little less than the temperature of the photons. Explain why they aren't at the same temperature, and why the photons are hotter.

They started at the same temperature, but there interactions got slow enough that they had difficulty staying in thermal equilibrium. Shortly thereafter, the electrons and positrons annihilated, increasing the temperature of the photons, but leaving the neutrinos cooler.

32. In the distant future of the universe, give the correct likely order for each of the following events: matter decays, last stars die, black holes evaporate, Local Group is isolated.

The order is: Local Group is isolated, last stars die, matter decays, black holes evaporate.

Part IV: Calculation (review material) [20/40 points] (20 points each)
PHY 310: Answer one of the two questions below
PHY 610: Answer both of the questions below

Group	m	d (kpc)	M
A	9.3	4.31	**–3.87**
B	12.4	18.0	**–3.88**
C	14.8	57.0	**–3.98**
D	19.3	**439**	**–3.91**

33. The brightest red-giant stars (as measured in the infrared) for four groups of stars are measured, and their distance is found by other methods. The results appear at right.

(a) For groups A, B, and C, find the absolute magnitude *M*.

We use the equation $m - M = 5\log d - 5$, making sure we work in pc (no kpc):

$$M_A = m_A - 5\log(d_A) + 5 = 9.3 - 5\log(4310) + 5 = -3.87$$
$$M_B = m_B - 5\log(d_B) + 5 = 12.4 - 5\log(18000) + 5 = -3.88$$
$$M_C = m_C - 5\log(d_C) + 5 = 14.8 - 5\log(57000) + 5 = -3.98$$

(b) Can we use the tip of the red-giant branch as a standard candle? Give an argument.

The fact that these numbers came out close (though not exactly the same) indicates that it is not a bad standard candle. The average value is about –3.91, so we will use that in part (c).

(c) The apparent magnitude of the brightest star group D is also measured, but its distance is unknown. Estimate the distance to group D.

Se assume $M = -3.91$, then we can just use the formula for the distance

$$d = 10^{1+\frac{m-M}{5}}\ \text{pc} = 10^{1+(19.3+3.91)/5}\ \text{pc} = 10^{5.642}\ \text{pc} = 4.39\times10^{5}\ \text{pc} = 439\ \text{kpc}.$$

Units
$1\ \text{pc} = 3.086\times10^{16}\ \text{m}$
$1\ \text{eV} = 1.602\times10^{-19}\ \text{J}$
$M_\odot = 1.989\times10^{30}\ \text{kg}$

Physical Constants
$k_B = 8.617\times10^{-5}\ \text{eV/K}$
$k_B = 1.381\times10^{-23}\ \text{J/K}$
$\hbar = 6.582\times10^{-16}\ \text{eV}\cdot\text{s}$
$\hbar c = 1.973\times10^{-7}\ \text{eV}\cdot\text{m}$

Magnitude/Distance
$m - M = 5\log d - 5$
$d = 10^{1+\frac{m-M}{5}}\ \text{pc}$

Age of Universe
Matter
$t = \dfrac{17.3\ \text{Gyr}}{(z+1)^{3/2}}$
Radiation
$t = \dfrac{2.42\ \text{s}}{\sqrt{g_{\text{eff}}}}\left(\dfrac{\text{MeV}}{k_B T}\right)^2$

Electron Mass
$mc^2 = 511\ \text{keV}$

Hubble's Constant
$H_0 \approx 67.8\ \text{km/s/Mpc}$

Mass Density
$\rho = g_{\text{eff}} \dfrac{\pi^2 (k_B T)^4}{30(\hbar c)^3 c^2}$
$g_{\text{eff},0} = 3.36$

34. The H-α line normally has a wavelength of 656.28 nm, but from the Hydra Cluster it is found to have a typical wavelength of about 692.24 nm.

(a) What is the red-shift *z* for this cluster?

By definition, $1+z=\lambda/\lambda_0$, so we have

$$z=\frac{\lambda}{\lambda_0}-1=\frac{692.24 \text{ nm}}{656.28 \text{ nm}}-1=1.0548-1=0.0548.$$

(b) What is the velocity of this cluster, and is it moving towards us or away from us? You may use the small *z* approximation for this calculation.

Since the red-shift is positive, the cluster is moving away from us. Since *z* is small, we estimate $z=v_r/c$, so

$$v_r=zc=0.0548\left(2.998\times10^{8} \text{ m/s}\right)=1.643\times10^{7} \text{ m/s}=16,430 \text{ km/s}.$$

(c) What is the approximate distance to the Hydra Cluster?

Hubble's law relates distance and velocity $v=H_0 d$, so we can simply solve for the distance:

$$d=\frac{v}{H_0}=\frac{16{,}430 \text{ km/s}}{68.7 \text{ km/s/Mpc}}=239 \text{ Mpc}.$$

(d) The angular size of the Hydra Cluster is about 45 arc-minutes. What is the approximate physical size of the Hydra Cluster?

We can use the formula $s=\alpha d$, where α is the angular size in radians. So we have

$$s=\alpha d=45'\cdot\frac{1^\circ}{60'}\cdot\frac{\pi \text{ rad}}{180^\circ}\left(239 \text{ Mpc}\right)=3.13 \text{ Mpc}.$$

Part V: Calculation (new material): [80/100 points] (20 points each)
PHY 310: Choose four of the following five questions and perform the calculations
PHY 610: Do all five of the following problems

35. The current density of baryons, cosmological constant, and cosmic background radiation are

$$\rho_{b0} = 4.196\times10^{-28}\ \text{kg/m}^3,\quad \rho_{\Lambda 0} = 5.966\times10^{-27}\ \text{kg/m}^3,\quad \rho_{r0} = 4.642\times10^{-31}\ \text{kg/m}^3$$

(a) Write a formula for these densities as a function of red shift z.

Baryons are matter, whose density is inversely proportional to the volume, so that $\rho_b \propto a^{-3}$. This means that $\rho_b a^3 = \rho_{b0} a_0^3$, and since $a_0/a = 1+z$, we have $a_0 = a(1+z)$ and hence $\rho_b a^3 = \rho_{b0} a^3 (1+z)^3$, or $\rho_b = \rho_{b0}(1+z)^3$. In a similar manner, since radiation satisfies $\rho_r \propto a^{-4}$ we can show that $\rho_r = \rho_{r0}(1+z)^4$. Finally, the cosmological constant does not scale at all, so we have

$$\rho_b = \rho_{b0}(1+z)^3,\quad \rho_\Lambda = \rho_{\Lambda 0},\quad \rho_r = \rho_{r0}(1+z)^4.$$

(b) At what red-shift *z* did the radiation density equal the cosmological constant density?

We simply equate $\rho_r = \rho_\Lambda$, so we have

$$\rho_{r0}(1+z)^4 = \rho_{\Lambda 0},$$
$$(1+z)^4 = \frac{\rho_{\Lambda 0}}{\rho_{r0}} = \frac{5.966\times10^{-27}\ \text{kg/m}^3}{4.642\times10^{-31}\ \text{kg/m}^3} = 1.285\times10^4,$$
$$z = \left(1.285\times10^4\right)^{1/4} - 1 = 9.647.$$

(c) What was the baryon density at this time?

We simply substitute the value of *z* into the formula for the baryon density, so we have

$$\rho_b = \rho_{b0}(1+z)^3 = \left(4.196\times10^{-28}\ \text{kg/m}^3\right)(10.647)^3 = 5.065\times10^{-25}\ \text{kg/m}^3.$$

(d) What dominated the universe at this time? Estimate the age of the universe at this time.

The baryon density at the time exceeds the cosmological constant, which also equals the radiation density. We conclude that matter must be dominating. We then use the formula for the age of the universe in the matter-dominated era:

$$t = \frac{17.3\ \text{Gyr}}{(z+1)^{3/2}} = \frac{17.3\ \text{Gyr}}{(10.647)^{1.5}} = 0.498\ \text{Gyr} = 498\ \text{Myr}.$$

36. The cosmic microwave background (CMBR) has a temperature of T_0 = 2.725 K.

(a) What is the typical energy of a photon at this temperature in eV?

A typical energy is given by $\bar{E} = 3k_B T$, so we have

$$\bar{E} = 3k_B T = 3\left(8.617\times10^{-5}\ \text{eV/K}\right)\left(2.725\ \text{K}\right) = 7.045\times10^{-4}\ \text{eV}.$$

(b) If two photons collide head on, they can produce an e^+e^- pair if they have energy $E_1E_2 > \left(mc^2\right)^2$, where $mc^2 = 5.11\times10^5\ \text{eV}$ is the electron self-energy. What is the minimum energy of a photon E_2 from some distant source such that it could collide with a CMBR photon to produce an e^+e^- pair?

The CMBR photon with energy $E_1 = 7.045\times10^{-4}\ \text{eV}$ is going to collide with a photon with energy E_2. We will need

$$E_2 > \frac{\left(mc^2\right)^2}{E_1} = \frac{\left(5.11\times10^5\ \text{eV}\right)^2}{7.045\times10^{-4}\ \text{eV}} = 3.707\times10^{14}\ \text{eV}.$$

(c) The density of photons from a thermal distribution is given by $n = 0.244\left(k_B T/\hbar c\right)^3$. Find the density of photons today in m^{-3}.

This is just straightforward plug and chug. We have

$$n = 0.244\left(\frac{k_B T}{\hbar c}\right)^3 = 0.244\left(\frac{\frac{1}{3}\bar{E}}{\hbar c}\right)^3 = 0.244\left(\frac{7.045\times10^{-4}\ \text{eV}}{3\times1.973\times10^{-7}\ \text{eV}\cdot\text{m}}\right)^3 = 4.11\times10^8\ \text{m}^{-3}.$$

(d) The cross section for the process $\gamma\gamma \to e^+e^-$ is approximately $\sigma = 7.94\times10^{-30}\ \text{m}^2$. Find the rate Γ in s^{-1} at which a high energy photon collides with a CMBR photon, and the time Γ^{-1} between collisions.

We can only estimate, but we would have a relative velocity of $\Delta v \simeq 2c$, so we have

$$\Gamma = n\sigma\left(\Delta v\right) = \left(4.11\times10^8\ \text{m}^{-3}\right)\left(7.94\times10^{-30}\ \text{m}^2\right)\left(2\times2.998\times10^8\ \text{m/s}\right) = 1.96\times10^{-12}\ \text{s}^{-1},$$

$$\Gamma^{-1} = \left(1.96\times10^{-12}\ \text{s}^{-1}\right)^{-1} = 5.11\times10^{11}\ \text{s} = 16{,}000\ \text{yr}.$$

(e) What is the mean distance such a high-energy photon can go before colliding with a CMBR photon?

This is just velocity times time, so

$$d = ct = c\Gamma^{-1} = \left(2.998\times10^8\ \text{m/s}\right)\left(5.11\times10^{11}\ \text{s}\right) = \frac{1.532\times10^{20}\ \text{m}}{3.086\times10^{16}\ \text{m/pc}} = 4960\ \text{pc}.$$

37. In the two-Higgs model, the number of boson and fermion degrees of freedom are $g_b = 32$ and $g_f = 90$ respectively. In the minimal supersymmetric standard model, there will be an additional fermion for each boson, and boson for every fermion.

(a) For $k_B T > 10^7$ MeV, assume the universe is hot enough that all particles can be considered massless. What is g_{eff} in the two-Higgs model? The supersymmetric model?

We simply use the formula $g_{eff} = g_b + \frac{7}{8} g_f$. For the two Higgs model, these numbers are given. For supersymmetry, we add them together to get $g_b = g_f = 32 + 90 = 122$. So we have

$$g_{eff,2} = 32 + \tfrac{7}{8} 90 = 110.75, \quad g_{eff,S} = 122 + \tfrac{7}{8} 122 = 228.75.$$

(b) What would be the age of the universe in each of these models at $k_B T = 1.00 \times 10^8$ MeV?

At these temperatures, we use the radiation-era time formula, so we have

$$t_2 = \frac{2.42 \text{ s}}{\sqrt{g_{\text{eff,2}}}} \left(\frac{\text{MeV}}{k_B T} \right)^2 = \frac{2.42 \text{ s}}{\sqrt{110.75}} \left(\frac{\text{MeV}}{1.00 \times 10^8 \text{ MeV}} \right)^2 = 2.30 \times 10^{-17} \text{ s},$$

$$t_S = \frac{2.42 \text{ s}}{\sqrt{g_{\text{eff,S}}}} \left(\frac{\text{MeV}}{k_B T} \right)^2 = \frac{2.42 \text{ s}}{\sqrt{228.75}} \left(\frac{\text{MeV}}{1.00 \times 10^8 \text{ MeV}} \right)^2 = 1.60 \times 10^{-17} \text{ s}.$$

(c) What would be the temperature $k_B T$ of the universe in each of these models at $t = 1.00 \times 10^{-18}$ s?

We solve the equations for the relevant temperature, which gives us

$$\frac{k_B T}{\text{MeV}} = \sqrt{\frac{2.42 \text{ s}}{t \sqrt{g_{\text{eff,2}}}}}.$$

We then just apply it in each case.

$$k_B T_2 = \sqrt{\frac{2.42 \text{ s}}{t \sqrt{g_{\text{eff,2}}}}} \text{ MeV} = \sqrt{\frac{2.42 \text{ s}}{\left(1.00 \times 10^{-18} \text{ s}\right) \sqrt{110.75}}} \text{ MeV} = 4.80 \times 10^8 \text{ MeV} = 480 \text{ TeV},$$

$$k_B T_S = \sqrt{\frac{2.42 \text{ s}}{t \sqrt{g_{\text{eff,S}}}}} \text{ MeV} = \sqrt{\frac{2.42 \text{ s}}{\left(1.00 \times 10^{-18} \text{ s}\right) \sqrt{228.75}}} \text{ MeV} = 4.00 \times 10^8 \text{ MeV} = 400 \text{ TeV}.$$

38. The *average* density of matter baryons today is about $\rho_{m0} = 2.655\times10^{-27}\ \text{kg/m}^3$. The mass of our galaxy is probably around $10^{12}\, M_\odot$, and it has an approximate radius (including dark matter) of about $100\ \text{kpc}$. Assume the galaxy is a uniform density sphere.

(a) Find the density of our galaxy in kg/m³. Find the ratio of the galaxy density to the average density of the universe.

The density of the galaxy is mass divided by volume. The volume of a sphere is

$$V = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi\left[\left(100\times10^3\ \text{pc}\right)\left(3.086\times10^{16}\ \text{m}\right)\right]^3 = 1.23\times10^{65}\ \text{m}^3.$$

The density, therefore, is

$$\rho_{g0} = \frac{M}{V} = \frac{10^{12}\left(1.989\times10^{30}\ \text{kg}\right)}{1.23\times10^{65}\ \text{m}^3} = 1.62\times10^{-23}\ \text{kg/m}^3,$$

$$\frac{\rho_{g0}}{\rho_{m0}} = \frac{1.62\times10^{-23}\ \text{kg/m}^3}{2.655\times10^{-27}\ \text{kg/m}^3} = 6085.$$

(b) Assume the galaxy has stayed the same size and mass since it formed, while the rest of the universe was expanding. At what red-shift *z* was the average matter density of the universe comparable to the density of the galaxy?

When the universe was a factor of $1+z$ smaller, it was also a factor of $(1+z)^3$ denser. Hence the matter density at the time was $\rho_m = \rho_{m0}(1+z)^3$. Equating this to the galaxy density, we have $\rho_{g0} = \rho_m = \rho_{m0}(1+z)^3$, so

$$(1+z)^3 = \frac{\rho_{g0}}{\rho_{m0}} = 6085,$$

$$z = 6085^{1/3} - 1 = 17.26.$$

This is earlier than galaxies actually formed, but we are supposed to ignore this and keep going. What probably actually happened is that the two were comparable around $z = 9$ or so, but then the galaxy contracted by about a factor of two, while the rest of the universe expanded by a factor of ten.

(c) Estimate the age of the universe at this time.

This is well into the matter-dominated period, so we use the formula

$$t = \frac{17.3\ \text{Gyr}}{(z+1)^{3/2}} = \frac{17.3\ \text{Gyr}}{(18.257)^{1.5}} = 0.222\ \text{Gyr} = 222\ \text{Myr}.$$

39. The densest element, Osmium, has a mass density of $2.259\times10^4\ \text{kg/m}^3$. At some point, this was the density of everything in the universe. Assume that the universe is radiation-dominated at the time, and the effective number of spin degrees of freedom is the same now as it was then.

(a) Find the temperature k_BT of the universe at this time in eV.

We simply use the formula for the mass density and solve for the temperature

$$\rho = g_{\text{eff}} \frac{\pi^2 (k_BT)^4}{30(\hbar c)^3 c^2},$$

$$(k_BT)^4 = \frac{30\rho(\hbar c)^3 c^2}{\pi^2 g_{\text{eff}}} = \frac{30\left(2.259\times10^4\ \text{kg/m}^3\right)\left(1.973\times10^{-7}\ \text{eV}\cdot\text{m}\right)^3\left(2.998\times10^8\,\text{m/s}\right)^2}{\pi^2(3.36)}$$

$$= 14.11\ \text{eV}^3\cdot\text{J}\cdot\frac{1\ \text{eV}}{1.602\times10^{-19}\ \text{J}} = 8.81\times10^{19}\ \text{eV}^4\ ,$$

$$k_BT = \left(8.81\times10^{19}\ \text{eV}^4\right)^{1/4} = 9.69\times10^4\ \text{eV} = 96.9\ \text{keV} = 0.0969\ \text{MeV}\,.$$

(b) Find the age of the universe at this time.

We use the formula with the same value of g_{eff}:

$$t = \frac{2.42\ \text{s}}{\sqrt{g_{\text{eff}}}}\left(\frac{\text{MeV}}{k_BT}\right)^2 = \frac{2.42\ \text{s}}{\sqrt{3.36}}\left(\frac{\text{MeV}}{0.0969}\right)^2 = 141\ \text{s}\,.$$

(c) Is the universe radiation dominated at this time? Is the value of the number of spin degrees of freedom you used approximately accurate? Give an argument.

The universe is radiation dominated at temperatures above about 1 eV, so this is clearly in the radiation dominated era. The value of g_{eff} is valid back to the time of electron-positron annihilation, which occurs when $3k_BT = m_ec^2 = 511\ \text{keV}$, or $k_BT = 170\ \text{keV}$. Since the temperature is cooler than 170 keV (by about a factor of two), this is an appropriate value of g_{eff} to use.