The math of the Solow growth model

Allin Cottrell

Spring 2019

These notes build on the notes titled "The Solow Growth Model." There we saw that the Cobb–Douglas production function can be written in per-worker terms as

$$y = Ak^{\alpha} \qquad 0 < \alpha < 1 \tag{1}$$

where $y \equiv Y/N$, $k \equiv K/N$, and N is the number of workers.

We also saw that in the steady state, the amount of investment per worker (sy, with $0 < s < 1$) must equal the investment required to hold capital per worker constant, namely $(n + \delta)k$, where n is the rate of growth of the workforce and δ is the rate of depreciation (that is, the fraction of capital stock that must be replaced each year just to keep the capital stock from shrinking).

If any of the above is unclear to you, please re-read the prior set of notes carefully. Now we take this reasoning a few steps further.

The steady state requirement is that

$$sy = (n + \delta)k \tag{2}$$

Combining this with equation (1), we get:

$$\begin{aligned} sy = sAk^{\alpha} &= (n + \delta)\, k \\ \frac{k}{k^{\alpha}} &= \frac{sA}{n + \delta} \\ k^{1-\alpha} &= \frac{sA}{n + \delta} \end{aligned}$$

and therefore

$$k_{ss} = \left(\frac{sA}{n + \delta} \right)^{\frac{1}{1-\alpha}} \tag{3}$$

This gives us the steady-state level of capital per worker as a function of the parameters s, A, n and δ, enabling us to figure out how changes in the parameters will affect the steady state.

Note that since α is a fraction, the exponent $1/(1 - \alpha)$ will be positive (and greater than one). That means that steady-state capital per worker is raised by increases in s or A but reduced by increases in n or δ. And via equation (1) the level of output per worker in the steady state will move in the same direction as capital per worker.

Now consider consumption per worker, $c \equiv C/N$, where C is total real consumption. If the fraction s of income is saved (and invested), then the complementary fraction, $1 - s$, is consumed. So

$$c = (1 - s)y = (1 - s)Ak^{\alpha}$$

We know—see equation (3)—that higher s produces a higher steady-state value of k, which in turn raises steady-state income per worker. By itself, that would enable greater consumption per worker. On the other hand, higher s means that a smaller *fraction* of income is available for consumption. How do these effects balance out?

From the Solow model diagram (review your class notes), we can see that steady-state $c \equiv C/N$ is maximized when the slope of the y curve equals $n+\delta$ (that is, the slope of the line showing how much investment must take place to hold k steady). To find the slope of y, take the first derivative of equation (1):

$$\frac{dy}{dk} = \alpha A k^{\alpha-1}$$

Setting this equal to $n + \delta$ gives

$$\begin{aligned} \alpha A k^{\alpha-1} &= n + \delta \\ k^{\alpha-1} &= \frac{n+\delta}{\alpha A} \\ k^{1-\alpha} &= \frac{\alpha A}{n+\delta} \end{aligned}$$

And so the value of k which maximizes consumption per worker is

$$k_{\text{cmax}} = \left(\frac{\alpha A}{n+\delta}\right)^{\frac{1}{1-\alpha}} \tag{4}$$

Finally, compare equations (3) and (4): the first of these gives steady-state k as a function of s, while the second tells us what k must be to maximize consumption. These values agree if and only if $s = \alpha$.

It turns out that in order for an economy to have a steady-state in which consumption per worker is as great as possible it is necessary to have a saving rate, s, which equals the exponent of capital in the production function, α. If s is less than α it will be possible to raise steady-state consumption by saving more, but there comes a point beyond which a further increase in the saving rate *reduces* steady-state consumption per worker (even though it raises steady-state k and y).